

Some necessary and sufficient conditions for parastrophic invariance of the associative law in quasigroups^{*†}

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Abstract

Every quasigroup (S, \cdot) belongs to a set of 6 quasigroups, called parastrophes denoted by (S, π_i) , $i \in \{1, 2, 3, 4, 5, 6\}$. It is shown that isotopy-isomorphy is a necessary and sufficient condition for any two distinct quasigroups (S, π_i) and (S, π_j) , $i, j \in \{1, 2, 3, 4, 5, 6\}$ to be parastrophic invariant relative to the associative law. In addition, a necessary and sufficient condition for any two distinct quasigroups (S, π_i) and (S, π_j) , $i, j \in \{1, 2, 3, 4, 5, 6\}$ to be parastrophic invariant under the associative law is either if the π_i -parastrophe of H is equivalent to the π_i -parastrophe of the holomorph of the π_i -parastrophe of S or if the π_i -parastrophe of H is equivalent to the π_k -parastrophe of the π_i -parastrophe of the holomorph of the π_i -parastrophe of S , for a particular $k \in \{1, 2, 3, 4, 5, 6\}$.

1 Introduction

Let L be a non-empty set. Define a binary operation (\cdot) on L such that $x \cdot y \in L$ for all x, y in L . Then, (L, \cdot) is called a groupoid. If the system of equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions for x and y respectively, then (L, \cdot) is called a quasigroup. At times we write $y = x \backslash z$ or equivalently $x = z / y$ for $x \cdot y = z$. Furthermore, if there exists a unique element $e \in L$ called the identity element such that for all x in L , $x \cdot e = e \cdot x = x$, (L, \cdot) is called a loop. On the other hand if in the quasigroup (L, \cdot) , $xy \cdot z = x \cdot yz$ for all x, y, z in L (associativity property), then (L, \cdot) is called a group and has an identity element ([20]). So, a loop is not a group but a group is a loop. This is the foundational reason for the study of non-associative algebraic system

It has been noted that every quasigroup (L, \cdot) belongs to a set of 6 quasigroups, called adjugates by Fisher and Yates [15], conjugates by Stein [32], [31] and Belousov [3] and

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parastrophes by Sade [26]. They have been studied by Artzy [2], Charles Lindner and Dwight Steedley [21] and a detailed study on them can be found in [23], [9] and [11]. The most recent study of the parastrophes of a quasigroup(loop) are by Sokhatskii [29, 30], Duplak [12] and Shchukin and Gushan [28]. For a quasigroup (L, \cdot) , its parastrophes are denoted by (L, π_i) , $i \in \{1, 2, 3, 4, 5, 6\}$ hence one can take $(L, \cdot) = (L, \pi_1)$. A quasigroup which is equivalent to all its parastrophes is called a totally symmetric quasigroup(introduced by Bruck [5]) while its loop is called a Steiner loop. For more on quasigroup, loops and their properties, readers should check [23], [7], [9], [11], [16] and [33]. Let (G, \oplus) and (H, \otimes) be two distinct quasigroups. The triple (A, B, C) such that

$$A, B, C : (G, \oplus) \longrightarrow (H, \otimes)$$

are bijections is said to be an isotopism if and only if

$$xA \otimes yB = (x \oplus y)C \quad \forall x, y \in G.$$

Thus, H is called an isotope of G and they are said to be isotopic. According to [23], every group is a G-loop(i.e a loop that is isomorphic to all its loop isotopes). Hence, every loop isotope of a group is a group but this is not true for quasigroup isotopes of a group, they are not necessarily associative. The aim of this work is to find necessary and sufficient condition(s) for any parastrophe of a group to be a group.

Let $Aum(L, \theta)$ be the automorphism group of a loop(quasigroup) (L, θ) , and the set $H = Aum(L, \theta) \times (L, \theta)$. If we define ' \circ ' on H such that

$$(\alpha, x) \circ (\beta, y) = (\alpha\beta, x\beta\theta y) \quad \forall (\alpha, x), (\beta, y) \in H,$$

then $H(L, \theta) = (H, \circ)$ is a loop(quasigroup) according to Bruck [6] and is called the Holomorph of (L, θ) . It can be observed that a loop is a group if and only if its holomorph is a group.

Interestingly, Adeniran [1] and Robinson [24], Oyebo and Adeniran [22], Chiboka and Solarin [10], Bruck [6], Bruck and Paige [8], Robinson [25] and Huthnance [17] have respectively studied the holomorphs of Bruck loops and Bol loops, central loops, conjugacy closed loops, inverse property loops, A-loops, extra loops and weak inverse property loops.

In this paper, it is proved that (S, \cdot) is an associative quasigroup if and only if any one of four particular parastrophes of (S, \cdot) obeys a Khalil condition or Evans' generalized associative law or Belousov's balanced identity or Falconer's generalized group identity. It is shown that isotopy-isomorphy is a necessary and sufficient condition for any two distinct quasigroups (S, π_i) and (S, π_j) , $i, j \in \{1, 2, 3, 4, 5, 6\}$ to be parastrophic invariant relative to the associative law.

Furthermore, the relationship between the parastrophes of the holomorph of a quasigroup and the holomorphs of the parastrophes of the same quasigroup are investigated. The following results are proved. For a quasigroup L with holomorph H , it is shown that the π_i -parastrophe of H is equivalent to the π_i -parastrophe of the holomorph of the π_i -parastrophe of L if and only if L is equivalent to its π_i -parastrophe for each $i = 1, 2, 3, 4, 5, 6$.

A necessary and sufficient condition for any two distinct quasigroups (S, π_i) and (S, π_j) , $i, j \in \{1, 2, 3, 4, 5, 6\}$ to be parastrophic invariant under the associative law is either if the π_i -parastrophe of H is equivalent to the π_i -parastrophe of the holomorph of the π_i -parastrophe of S or if the π_i -parastrophe of H is equivalent to the π_k -parastrophe of the π_i -parastrophe of the holomorph of the π_i -parastrophe of S , for a particular $k \in \{1, 2, 3, 4, 5, 6\}$.

2 Preliminaries

Definition 2.1 *Let (L, θ) be a quasigroup. The 5 parastrophes or conjugates or adjugates of (L, θ) are quasigroups whose binary operations θ^* , θ^{-1} , ${}^{-1}\theta$, $(\theta^{-1})^*$, $({}^{-1}\theta)^*$ defined on L are given by :*

- (a) (L, θ^*) : $y\theta^*x = z \Leftrightarrow x\theta y = z \forall x, y, z \in L$.
- (b) (L, θ^{-1}) : $x\theta^{-1}z = y \Leftrightarrow x\theta y = z \forall x, y, z \in L$.
- (c) $(L, {}^{-1}\theta)$: $z{}^{-1}\theta y = x \Leftrightarrow x\theta y = z \forall x, y, z \in L$.
- (d) $(L, (\theta^{-1})^*)$: $z(\theta^{-1})^*x = y \Leftrightarrow x\theta y = z \forall x, y, z \in L$.
- (e) $(L, ({}^{-1}\theta)^*)$: $y({}^{-1}\theta)^*z = x \Leftrightarrow x\theta y = z \forall x, y, z \in L$.

The five parastrophes of $H(L, \theta) = (H, \circ)$ shall be denoted by

$$(H, \circ^*), (H, \circ^{-1}), (H, {}^{-1}\circ), (H, (\circ^{-1})^*), \text{ and } (H, ({}^{-1}\circ)^*).$$

The holomorphes of the five parastrophes of the quasigroup (L, θ) are denoted by :

$$H(L, \theta^*) = (H, \circ_*), H(L, \theta^{-1}) = (H, \circ_{-1}), H(L, {}^{-1}\theta) = (H, {}_{-1}\circ),$$

$$H(L, (\theta^{-1})^*) = (H, (\circ_{-1})_*) \text{ and } H(L, ({}^{-1}\theta)^*) = (H, {}_{*}({}_{-1}\circ)).$$

Definition 2.2 *Let (L, θ) be a quasigroup.*

- (a) R_x and L_x represent the right and left translation maps in $(L, \theta) \forall x \in L$.
- (b) R_x^* and L_x^* represent the right and left translation maps in $(L, \theta^*) \forall x \in L$.
- (c) \mathcal{R}_x and \mathcal{L}_x represent the right and left translation maps in $(L, \theta^{-1}) \forall x \in L$.
- (d) \mathbb{R}_x and \mathbb{L}_x represent the right and left translation maps in $(L, {}^{-1}\theta) \forall x \in L$.
- (e) \mathcal{R}_x^* and \mathcal{L}_x^* represent the right and left translation maps in $(L, (\theta^{-1})^*) \forall x \in L$.
- (f) \mathbb{R}_x^* and \mathbb{L}_x^* represent the right and left translation maps in $(L, ({}^{-1}\theta)^*) \forall x \in L$.

Remark 2.1 If (L, θ) is a loop, (L, θ^*) is also a loop (and vice versa) while the other adjugates are quasigroups. Furthermore, (L, θ^{-1}) and $(L, (-^1\theta)^*)$ have left identity elements, that is they are left loops while $(L, {}^{-1}\theta)$ and $(L, (\theta^{-1})^*)$ have right identity elements, that is they are right loops. (L, θ^{-1}) or $(L, {}^{-1}\theta)$ or $(L, (\theta^{-1})^*)$ or $(L, (-^1\theta)^*)$ is a loop if and only if (L, θ) is a loop of exponent 2.

Lemma 2.1 If (L, θ) is a quasigroup, then

1. $R_x^* = L_x$, $L_x^* = R_x$, $\mathcal{L}_x = L_x^{-1}$, $\mathbb{R}_x = R_x^{-1}$, $\mathcal{R}_x^* = L_x^{-1}$, $\mathbb{L}_x^* = R_x^{-1} \forall x \in L$.
2. $\mathcal{L}_x = R_x^{*-1}$, $\mathbb{R}_x = L_x^{*-1}$, $\mathcal{R}_x^* = R_x^{*-1} = \mathcal{L}_x$, $\mathbb{L}_x^* = L_x^{*-1} = \mathbb{R}_x \forall x \in L$.

Proof

The proof of these follows by using Definition 2.1 and Definition 2.2.

1. $y\theta^*x = z \Leftrightarrow x\theta y = z \Rightarrow y\theta^*x = x\theta y \Rightarrow yR_x^* = yL_x \Rightarrow R_x^* = L_x$. Also, $y\theta^*x = x\theta y \Rightarrow xL_y^* = xR_y \Rightarrow L_y^* = R_y$.
 $x\theta^{-1}z = y \Leftrightarrow x\theta y = z \Rightarrow x\theta(x\theta^{-1}z) = z \Rightarrow x\theta z\mathcal{L}_x = z \Rightarrow z\mathcal{L}_xL_x = z \Rightarrow \mathcal{L}_xL_x = I$.
Also, $x\theta^{-1}(x\theta y) = y \Rightarrow x\theta^{-1}yL_x = y \Rightarrow yL_x\mathcal{L}_x = y \Rightarrow L_x\mathcal{L}_x = I$. Hence, $\mathcal{L}_x = L_x^{-1} \forall x \in L$.
 $z(-^1\theta)y = x \Leftrightarrow x\theta y = z \Rightarrow (x\theta y)(-^1\theta)y = x \Rightarrow xR_y(-^1\theta)y = x \Rightarrow xR_y\mathbb{R}_y = x \Rightarrow R_y\mathbb{R}_y = I$. Also, $(z(-^1\theta)y)\theta y = z \Rightarrow z\mathbb{R}_y\theta y = z \Rightarrow z\mathbb{R}_yR_y = z \Rightarrow \mathbb{R}_yR_y = I$.
Thence, $\mathbb{R}_y = R_y^{-1} \forall x \in L$.
 $z(\theta^{-1})^*x = y \Leftrightarrow x\theta y = z$, so, $x\theta(z(\theta^{-1})^*x) = z \Rightarrow x\theta z\mathcal{R}_x^* = z \Rightarrow z\mathcal{R}_x^*L_x = z \Rightarrow \mathcal{R}_x^*L_x = I$. Also, $(x\theta y)(\theta^{-1})^*x = y \Rightarrow yL_x(\theta^{-1})^*x = y \Rightarrow yL_x\mathcal{R}_x^* = y \Rightarrow L_y\mathcal{R}_x^* = I$.
Whence, $\mathcal{R}_x^* = L_x^{-1}$.
 $y(-^1\theta)^*z = x \Leftrightarrow x\theta y = z$, so, $y(-^1\theta)^*(x\theta y) = x \Rightarrow y(-^1\theta)^*xR_y = x \Rightarrow xR_y\mathbb{L}_y^* = x \Rightarrow R_y\mathbb{L}_y^* = I$. Also, $(y(-^1\theta)^*z)\theta y = z \Rightarrow z\mathbb{L}_y^*\theta y = z \Rightarrow z\mathbb{L}_y^*R_y = z \Rightarrow \mathbb{L}_y^*R_y = I$. Thus, $\mathbb{L}_y^* = R_y^{-1}$.
2. These ones follow from 1..

Theorem 2.1 (Falconer Theorem 2.9 [14])

If a quasigroup Q is isotopic to a group G , then all the parastrophes of Q are isotopic to G .

In the past, some isotopy closure properties for groups (i.e necessary and sufficient conditions for a quasigroup to be isotopic to a group) and isotopy-isomorphy conditions for groups (i.e necessary and sufficient conditions for the isomorphism of quasigroups isotopic to a group [18]) have been proved. Some of the formal are stated below.

Theorem 2.2 (Evans [13])

A quasigroup (Q, \cdot) is isotopic to a group if and only if Q obeys the law

$$[(xP_1 \cdot yP_2)P_3 \cdot zP_4]P_5 = [xQ_1 \cdot (yQ_2 \cdot zQ_3)Q_4]Q_5 \quad \text{Evans law}$$

where P_i, Q_i , $i = 1, 2, 3, 4, 5$ are permutations on (Q, \cdot) .

Theorem 2.3 (Belousov [4])

A quasigroup (Q, \cdot) is isotopic to a group if and only if Q obeys an identity

$$w_1(x_1, x_2, \dots, x_n) = w_2(x_1, x_2, \dots, x_n) \quad \text{balanced identity}$$

where each x_i , $i = 1, 2, \dots, n$, occurs exactly once in w_1 and in w_2 and w_1 and w_2 involve different grouping of some triple.

Theorem 2.4 (Falconer Theorem 2.10 [14])

A quasigroup (Q, \cdot) is isotopic to a group if and only if Q satisfies a generalized group identity

Theorem 2.5 (Khalil Conditions [27])

A quasigroup is an isotope of a group if and only if any one of the following six identities are true in the quasigroup for all elements x, y, z, u, v .

1. $x\{z \setminus [(z/u)v]\} = \{x(z \setminus z)/u\}v$
2. $x\{u \setminus [(z/u)v]\} = \{x(u \setminus z)/u\}v$
3. $x\{z \setminus [(u/u)v]\} = \{x(z \setminus u)/u\}v$
4. $x[y \setminus \{[(yy)/z]u\}] = [\{x[y \setminus (yy)]\}/z]u$
5. $x[y \setminus \{[(yz)/y]u\}] = [\{x[y \setminus (yz)]\}/y]u$
6. $x[z \setminus \{[(yy)/y]u\}] = [\{x[z \setminus (yy)]\}/y]u$

3 Main Results

3.1 Parastrophic invariance of groups

Theorem 3.1 Let G be a loop with identity element e and H a quasigroup such that they are isotopic under the triple $\alpha = (A, B, C)$.

1. If $C = B$, then $G \stackrel{A}{\cong} H$ if and only if $eB \in N_\rho(H)$ where $N_\rho(H)$ is the right nucleus of H .
2. If $C = A$, then $G \stackrel{B}{\cong} H$ if and only if $eA \in N_\lambda(H)$ where $N_\lambda(H)$ is the left nucleus of H .

Proof

Here, when L_x and R_x are respectively the left and right translations of the loop G then the left and right translations of its quasigroup isotope H are denoted by L'_x and R'_x respectively.

Let (G, \cdot) and (H, \circ) be any two distinct quasigroups. If $A, B, C : G \rightarrow H$ are permutations, then the following statements are equivalent :

- the triple $\alpha = (A, B, C)$ is an isotopism of G upon H .
- $R'_{xB} = A^{-1}R_xC \ \forall \ x \in G$.
- $L'_{yA} = B^{-1}L_yC \ \forall \ y \in G$.

1. When $\alpha = (A, B, B)$, $R'_{eB} = A^{-1}B \Rightarrow B = AR'_{eB}$. So,

$$\alpha = (A, AR'_{eB}, AR'_{eB}) = (A, A, A)(I, R'_{eB}, R'_{eB}), \ \alpha : G \rightarrow H.$$

If $(A, A, A) : G \rightarrow H$ is an isotopism i.e A is an isomorphism, then $(I, R'_{eB}, R'_{eB}) : H \rightarrow H$ is an autotopism if and only if $eB \in N_\rho(H)$.

2. When $\alpha = (A, B, A)$, $L'_{eA} = B^{-1}A \Rightarrow A = BL'_{eA}$. So,

$$\alpha = (BL'_{eA}, B, BL'_{eA}) = (B, B, B)(L'_{eA}, I, L'_{eA}), \ \alpha : G \rightarrow H.$$

If $(B, B, B) : G \rightarrow H$ is an isotopism i.e B is an isomorphism, then $(L'_{eA}, I, L'_{eA}) : H \rightarrow H$ is an autotopism if and only if $eA \in N_\lambda(H)$.

Theorem 3.2 *A quasigroup (S, θ) is associative if and only if any of the following equivalent statements is true.*

1. (S, θ) is isotopic to $(S, (\theta^{-1})^*)$. Hence, the other 4 parastrophes are also isotopic to $(S, (\theta^{-1})^*)$.
2. (S, θ^*) is isotopic to (S, θ^{-1}) . Hence, the other 4 parastrophes are also isotopic to (S, θ^{-1}) .
3. (S, θ) is isotopic to $(S, (-^1\theta)^*)$. Hence, the other 4 parastrophes are also isotopic to $(S, (-^1\theta)^*)$.
4. (S, θ^*) is isotopic to $(S, -^1\theta)$. Hence, the other 4 parastrophes are also isotopic to $(S, -^1\theta)$.

Proof

(S, θ) is associative if and only if $s_1\theta(s_2\theta s_3) = (s_1\theta s_2)\theta s_3 \Leftrightarrow R_{s_2}R_{s_3} = R_{s_2\theta s_3} \Leftrightarrow L_{s_1\theta s_2} = L_{s_2}L_{s_1} \ \forall \ s_1, s_2, s_3 \in S$.

The proof of the equivalence of 1. and 2. is as follows. $L_{s_1\theta s_2} = L_{s_2}L_{s_1} \Leftrightarrow \mathcal{L}_{s_1\theta s_2}^{-1} = \mathcal{L}_{s_2}^{-1}\mathcal{L}_{s_1}^{-1} \Leftrightarrow \mathcal{L}_{s_1\theta s_2} = \mathcal{L}_{s_1}\mathcal{L}_{s_2} \Leftrightarrow (s_1\theta s_2)\theta^{-1}s_3 = s_2\theta^{-1}(s_1\theta^{-1}s_3) \Leftrightarrow (s_1\theta s_2)\mathcal{R}_{s_3} = s_2\theta^{-1}s_1\mathcal{R}_{s_3} = s_1\mathcal{R}_{s_3}(\theta^{-1})^*s_2 \Leftrightarrow (s_1\theta s_2)\mathcal{R}_{s_3} = s_1\mathcal{R}_{s_3}(\theta^{-1})^*s_2 \Leftrightarrow (s_2\theta^*s_1)\mathcal{R}_{s_3} = s_2\theta^{-1}s_1\mathcal{R}_{s_3} \Leftrightarrow$

$$(\mathcal{R}_{s_3}, I, \mathcal{R}_{s_3}) : (S, \theta) \rightarrow (S, (\theta^{-1})^*) \Leftrightarrow \quad (1)$$

$$(I, \mathcal{R}_{s_3}, \mathcal{R}_{s_3}) : (S, \theta^*) \rightarrow (S, \theta^{-1}) \quad (2)$$

$\Leftrightarrow (S, \theta)$ is isotopic to $(S, (\theta^{-1})^*) \Leftrightarrow (S, \theta^*)$ is isotopic to (S, θ^{-1}) .

The proof of the equivalence of 3. and 4. is as follows. $R_{s_2}R_{s_3} = R_{s_2\theta s_3} \Leftrightarrow R_{s_2}^{-1}R_{s_3}^{-1} = R_{s_2\theta s_3}^{-1} \Leftrightarrow R_{s_3}R_{s_2} = R_{s_2\theta s_3} \Leftrightarrow (s_1^{-1}\theta s_3)^{-1}\theta s_2 = s_1^{-1}\theta(s_2\theta s_3) \Leftrightarrow (s_2\theta s_3)\mathbb{L}_{s_1} = s_3\mathbb{L}_{s_1}^{-1}\theta s_2 = s_2(-^1\theta)^*s_3\mathbb{L}_{s_1} \Leftrightarrow (s_2\theta s_3)\mathbb{L}_{s_1} = s_2(-^1\theta)^*s_3\mathbb{L}_{s_1} \Leftrightarrow (s_3\theta^*s_2)\mathbb{L}_{s_1} = s_3\mathbb{L}_{s_1}^{-1}\theta s_2 \Leftrightarrow$

$$(I, \mathbb{L}_{s_1}, \mathbb{L}_{s_1}) : (S, \theta) \rightarrow (S, (-^1\theta)^*) \Leftrightarrow \quad (3)$$

$$(\mathbb{L}_{s_1}, I, \mathbb{L}_{s_1}) : (S, \theta^*) \rightarrow (S, -^1\theta) \quad (4)$$

$\Leftrightarrow (S, \theta)$ is isotopic to $(S, (-^1\theta)^*) \Leftrightarrow (S, \theta^*)$ is isotopic to $(S, -^1\theta)$.

The last part of 1. to 4. follow by Theorem 2.1.

Remark 3.1 *In the proof of Theorem 3.2, it can be observed that the isotopisms are triples of the forms (A, I, A) and (I, B, B) . Weak associative identities such as the Bol, Moufang and extra identities have been found to be isotopic invariant in loops for any triple of the form (A, B, C) while in [19], the central identities have been found to be isotopic invariant only under triples of the forms (A, B, A) and (A, B, B) . Since associativity obeys all the Bol-Moufang identities, the observation in the theorem agrees with the latter stated facts.*

Corollary 3.1 *(S, θ) is an associative quasigroup if and only if any one of particular four parastrophes of (S, θ) obeys any of the six Khalil conditions or Evans law or a balanced identity or a generalized group identity.*

Proof

Let (S, θ) be the quasigroup in consideration. By hypothesis, (S, θ) is a group. Notice that $R_{s_2}R_{s_3} = R_{s_2\theta s_3} \Leftrightarrow L_{s_2\theta s_3}^* = L_{s_3}^*L_{s_2}^*$. Hence, (S, θ^*) is also a group. In Theorem 3.2, two of the parastrophes are isotopes of (S, θ) while the other two are isotopes of (S, θ^*) . Since the Khalil conditions, Evans law, balanced identity and generalized group identity of Theorem 2.5, Theorem 2.2, Theorem 2.3 and Theorem 2.4 respectively, are necessary and sufficient conditions for a quasigroup to be an isotope of a group, then they must be necessarily and sufficiently true in the four quasigroup parastrophes of (S, θ) .

Corollary 3.2 *Let (S, θ) be an associative quasigroup.*

1. $(S, \theta) \equiv (S, (\theta^{-1})^*)$ if and only if $(S, (\theta^{-1})^*)$ is associative.
2. $(S, \theta^*) \equiv (S, \theta^{-1})$ if and only if (S, θ^{-1}) is associative.
3. $(S, \theta) \equiv (S, (-^1\theta)^*)$ if and only if $(S, (-^1\theta)^*)$ is associative.
4. $(S, \theta^*) \equiv (S, -^1\theta)$ if and only if $(S, -^1\theta)$ is associative.

Proof

Let (S, θ) be an associative quasigroup with identity element e .

1. Using isotopism (1) of Theorem 3.2 in the second part of Theorem 3.1, it will be observed that $(S, \theta) \stackrel{I}{\cong} (S, (\theta^{-1})^*)$ if and only if $s(\theta^{-1})^*e = s \in N_\lambda(S, (\theta^{-1})^*) \forall s \in S$ if and only if $(S, (\theta^{-1})^*)$ is associative.

2. Using isotopism (2) of Theorem 3.2 in the first part of Theorem 3.1, it will be observed that $(S, \theta^*) \stackrel{I}{\cong} (S, \theta^{-1})$ if and only if $e\theta^{-1}s = s \in N_\rho(S, \theta^{-1}) \forall s \in S$ if and only if (S, θ^{-1}) is associative.
3. Using isotopism (3) of Theorem 3.2 in the first part of Theorem 3.1, it will be observed that $(S, \theta) \stackrel{I}{\cong} (S, (-^1\theta)^*)$ if and only if $e(-^1\theta)^*s = s \in N_\rho(S, (-^1\theta)^*) \forall s \in S$ if and only if $(S, (-^1\theta)^*)$ is associative.
4. Using isotopism (4) of Theorem 3.2 in the second part of Theorem 3.1, it will be observed that $(S, \theta^*) \stackrel{I}{\cong} (S, -^1\theta)$ if and only if $s(-^1\theta)e = s \in N_\lambda(S, -^1\theta) \forall s \in S$ if and only if $(S, -^1\theta)$ is associative.

Corollary 3.3 *Isotopy-isomorphy is a necessary and sufficient condition for any two distinct quasigroups (S, π_i) and (S, π_j) , $i, j \in \{1, 2, 3, 4, 5, 6\}$ to be parastrophic invariant under the associative law.*

Proof

By Theorem 3.2 and Corollary 3.2, the claim follows.

3.2 Parastrophy-Holomorphy and Holomorphy-Parastrophy of Quasigroups

Theorem 3.3 *Let (L, θ) be a quasigroup with holomorph $H(L, \theta) = (H, \circ)$. The following are true.*

1. $(H, \circ^*) \equiv (H, (\circ_*)^*) \Leftrightarrow (L, \theta) \equiv (L, \theta^*)$.
2. $(H, \circ^{-1}) \equiv (H, (\circ_{-1})^{-1}) \Leftrightarrow (L, \theta) \equiv (L, \theta^{-1})$.
3. $(H, -^1\circ) \equiv (H, -^1(\circ_{-1})) \Leftrightarrow (L, \theta) \equiv (L, -^1\theta)$.
4. $(H, (\circ^{-1})^*) \equiv (H, (((\circ_{-1})_*)^{-1})^*) \Leftrightarrow (L, \theta) \equiv (L, (\theta^{-1})^*)$.
5. $(H, (-^1\circ)^*) \equiv (H, (-^1((\circ_{-1})_*))^*) \Leftrightarrow (L, \theta) \equiv (L, (-^1\theta)^*)$.

Proof

$(H, \circ) : (\alpha, x) \circ (\beta, y) = (\alpha\beta, x\beta\theta y)$.

1. $(H, \circ^*) : (\beta, y) \circ (\alpha, x) = (\alpha\beta, x\beta\theta y) = (\alpha\beta, y\theta^*x\beta)$. $(H, \circ_*) : (\alpha, x) \circ_* (\beta, y) = (\alpha\beta, x\beta\theta^*y) = (\alpha\beta, y\theta x\beta) \Rightarrow (\beta, y)(\circ_*)^*(\alpha, x) = (\alpha\beta, x\beta\theta^*y) = (\alpha\beta, y\theta x\beta)$. Hence, $(H, \circ^*) \equiv (H, (\circ_*)^*) \Leftrightarrow (L, \theta) \equiv (L, \theta^*)$.
2. $(H, \circ^{-1}) : (\alpha, x) \circ^{-1}(\alpha\beta, x\beta\theta y) = (\beta, y)$. $(H, \circ_{-1}) : (\alpha, x) \circ_{-1}(\beta, y) = (\alpha\beta, x\beta\theta^{-1}y) \Rightarrow (\alpha, x)(\circ_{-1})^{-1}(\alpha\beta, x\beta\theta^{-1}y) = (\beta, y)$. Thence, $(H, \circ^{-1}) \equiv (H, (\circ_{-1})^{-1}) \Leftrightarrow (L, \theta) \equiv (L, \theta^{-1})$.

3. $(H, {}^{-1}\circ) : (\alpha\beta, x\beta\theta y)^{-1}\circ(\beta, y) = (\alpha, x)$. $(H, {}_{-1}\circ) : (\alpha, x)_{-1}\circ(\beta, y) = (\alpha\beta, x\beta^{-1}\theta y) \Rightarrow (\alpha\beta, x\beta^{-1}\theta y)^{-1}({}^{-1}\circ)(\beta, y) = (\alpha, x)$. Whence, $(H, {}^{-1}\circ) \equiv (H, {}^{-1}({}_{-1}\circ)) \Leftrightarrow (L, \theta) \equiv (L, {}^{-1}\theta)$.
4. $(H, (\circ^{-1})^*) : (\alpha\beta, x\beta\theta y)(\circ^{-1})^*(\alpha, x) = (\beta, y)$. $(H, (\circ_{-1})_*) : (\alpha, x)(\circ_{-1})_*(\beta, y) = (\alpha\beta, x\beta(\theta^{-1})^*y) \Rightarrow (\alpha\beta, x\beta(\theta^{-1})^*y)((\circ_{-1})_*)^{-1}(\alpha, x) = (\beta, y)$. Then, $(H, (\circ^{-1})^*) \equiv (H, (((\circ_{-1})_*)^{-1})^*) \Leftrightarrow (L, \theta) \equiv (L, (\theta^{-1})^*)$.
5. $(H, ({}^{-1}\circ)^*) : (\beta, y)({}^{-1}\circ)^*(\alpha\beta, x\beta\theta y) = (\alpha, x)$. $(H, ({}_{-1}\circ)_*) : (\alpha, x)({}_{-1}\circ)_*(\beta, y) = (\alpha\beta, x\beta({}^{-1}\theta)^*y) \Rightarrow (\beta, y)(({}_{-1}\circ)_*)^{-1}(\alpha\beta, x\beta\theta y) = (\alpha, x)$. So, $(H, ({}^{-1}\circ)^*) \equiv (H, ({}^{-1}({}_{-1}\circ)_*)^*) \Leftrightarrow (L, \theta) \equiv (L, ({}^{-1}\theta)^*)$.

Corollary 3.4 *Let the quasigroup (L, θ) be a group with holomorph $H(L, \theta) = (H, \circ)$. The following are true.*

1. $(H, \circ^{-1}) \equiv (H, ((\circ_{-1})^{-1})^*)$ if and only if (L, θ^{-1}) is associative.
2. $(H, {}^{-1}\circ) \equiv (H, ({}^{-1}({}_{-1}\circ))^*)$ if and only if $(L, {}^{-1}\theta)$ is associative.
3. $(H, (\circ^{-1})^*) \equiv (H, (((\circ_{-1})_*)^{-1})^*)$ if and only if $(L, (\theta^{-1})^*)$ is associative.
4. $(H, ({}^{-1}\circ)^*) \equiv (H, ({}^{-1}({}_{-1}\circ)_*)^*)$ if and only if $(L, ({}^{-1}\theta)^*)$ is associative.

Proof

1. to 4. are proved by applying 4. and 5. of Theorem 3.3 to 1. to 4. of Corollary 3.2.

Corollary 3.5 *A necessary and sufficient condition for any two distinct quasigroups (S, π_i) and (S, π_j) , $i, j \in \{1, 2, 3, 4, 5, 6\}$ to be parastrophic invariant under the associative law is either*

1. *if the π_i -parastrophe of H is equivalent to the π_i -parastrophe of the holomorph of the π_i -parastrophe of S or*
2. *if the π_i -parastrophe of H is equivalent to the π_k -parastrophe of the π_i -parastrophe of the holomorph of the π_i -parastrophe of S , for a particular $k \in \{1, 2, 3, 4, 5, 6\}$.*

Proof

The proofs of 1. and 2. can be deduced from Theorem 3.3 and Corollary 3.4 respectively.

Further Study It will be interesting to investigate parastrophic invariance in quasigroups relative to weak associative laws such as the Bol-Moufang identities and establish necessary and sufficient condition(s) for them to be parastrophic invariant in quasigroups.

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